The Ramanujan Cubes (Sorry Hardy, 1729 is Hardly a Story …)

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Abstract

There is a story of G.H. Hardy visiting his friend Srinivasa Ramanujan while the later was hospitalized and offering his friend the comment that he arrived in cab 1729. Lacking any topic of conversation Hardy made mention that the number 1729 was "uninteresting." Ramanujan's rejoinder that 1729 being the smallest sum of two cubes that can be expressed in two ways is now legendary.

Introduction

For such eccentric souls as the Cambridge mathematician G.H. Hardy and such remarkable compatriots as the Indian Srinivasa Ramanujan, there are stories of legendary intellectual prowess. Ramanujan who grew up in a hot climate never took to the coldness and dampness of England and was quite ill, quite often sometimes requiring hospitalization. In the hospital, far from other distractions, he did good mathematics.

In his autobiography *A Mathematician's Apology* (1940) Hardy tells the story of once visiting his friend in the hospital and offering his friend the comment that he arrived in cab 1729. Lacking any topic of conversation Hardy made mention that the number 1729 was "uninteresting."

Ramanujan's rejoinder was almost immediate – that 1729 being the smallest sum of two cubes that can be expressed in two ways is now legendary.

Here is how the story if presented by C.P. Snow who wrote a preface to Hardy's Book:

> "Hardy use to visit him, as he lay dying in hospital in Putney. It was on one of these visits that there happen the incident of the taxi-cab number. Hardy had gone out to Putney by taxi, as usual his chosen method of conveyance. He went into the room where Ramanunjan was lying.
>
> Hardy always inept at introducing a conversation, said, probably without a greeting, and certainly as his first remark, "I thought the number of my taxicab was 1729. It seemed to me a rather dull number."
>
> To which Ramanunjan replied "No Hardy! No Hardy!" It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways." [1]

When I first read of this story as a young boy I was awestruck. Now with a few years of mathematics under my belt I have to admit I am less so.

1728 – A Cube

Lets try to reconstruct his reasoning. Since Ramanujan worked extensively with numbers he knew that 1728 was a perfect cube. It is in fact straight forward to show that 1728 is a perfect cube. For the sake of argument assume that Ramanujan did not know and worked this out in his head. The number is marginally greater than 1000 where $1000 = 10^3$ from which we realize that $11^3 = 1331$ and

$$12^3 = 1728$$

You can in fact quite easily calculate this in your head

$$1728 = 12(12^2) = 12(144) = (10+2)144 = 1440 + 288$$

1729 – The Sum of Two Cubes

From this we easily find that

$$1729 = 12^3 + 1^3$$

How he arrived at the fact that it can be expressed in a second fashion as a sum of cubes is straightforward as well, namely using *Pascal's Triangle* it is easily shown that

$$(12^3) + 1^3 = (10 + 2)^3 + 1^3 = 1000 + 728 + 1 = 10^3 + 9^3$$

The Smallest Sum of Two Cubes

Since we have only to now look at, at most twelve numbers, it takes just a matter of a few seconds to realize that for the twelve cubes 1^3 to 12^3 there are no other pairs that can be expressed in two ways (this is left as a mental exercise to the reader) except for 1729. Let us list them out

$$1^3 = 1$$
$$2^3 = 8$$
$$3^3 = 27$$
$$4^3 = 64$$
$$5^3 = 125$$
$$6^3 = 216$$
$$7^3 = 343$$
$$8^3 = 512$$
$$9^3 = 729$$
$$10^3 = 1000$$
$$11^3 = 1331$$
$$12^3 = 1728$$

Ramanujan – A Prepared Mind States the Obvious

You may immediately note the two adjacent numbers in the list 729 and 1000. For someone like Ramanujan whose brain was wired for mathematics, his was a prepared mind that stated the obvious.

One also needs to remember that tables of mathematical functions were readily available during Ramanujan's lifetime and he used them extensively in his mathematical research (most good mathematicians still do).

It is said that Ramanujan had a sharp mind and a photographic memory. Mathematics was his life.

Sorry Hardy 1729 is hardly a story …

[1] Excerpt from the Preface by C.P. Snow to G. H. Hardy's *A Mathematician's Apology*, Cambridge University Press p. 37, 1967